

Dissipativity of the delay semigroup

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Abstract

It is shown that under mild conditions a delay semigroup can be transformed into a (generalized) contraction semigroup by changing the inner product on the (Hilbert) state space into an equivalent inner product. Applications in the field of stochastic evolution equations with delay are given as example.

Consider a linear functional evolution equation of the form

$$\begin{cases} \frac{d}{dt}u(t) = Bu(t) + \int_{-r}^0 d\zeta(\sigma) u(t + \sigma), & t \geq 0, \\ u(0) = x \\ u(\sigma) = f(\sigma), & -1 \leq \sigma \leq 0 \end{cases} \quad (1)$$

in a Hilbert space H where $\zeta : [-1, 0] \rightarrow L(H)$ of bounded variation, with initial condition $x \in H$ and $f \in L^2([-1, 0]; H)$. Under conditions on B and ζ there exists a unique solution to (1), see [1]. If we let $\mathcal{E}^2 = H \times L^2([-1, 0]; H)$ denote the state space of this functional evolution equation then we may consider the solution semigroup of this linear evolution:

$$T(t) \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} u(t) \\ u(t + \cdot) \end{pmatrix}. \quad (2)$$

We call $(T(t))_{t \geq 0}$ an abstract delay semigroup. This notion is introduced in more detail in Section 1.

The semigroup thus obtained is rarely a contraction semigroup. In this paper we will show that under very mild conditions, we may change the inner product on \mathcal{E}^2 into an equivalent inner product, such that the delay semigroup becomes a (generalized) contraction semigroup. A *contraction semigroup* is a semigroup $(S(t))_{t \geq 0}$ such that $\|S(t)\| \leq 1$ for all $t \geq 0$. A *generalized contraction semigroup* is a semigroup $(S(t))_{t \geq 0}$ such that $\|S(t)\| \leq e^{\lambda t}$ for some $\lambda \in \mathbb{R}$ and all $t \geq 0$.

Our result provides a generalisation of Section 10.3 in [5]. There a similar result was established for linear delay equations of the form

$$\frac{d}{dt}x(t) = Bx(t) + \sum_{i=1}^k B_i x(t - h_i).$$

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The conditions established there are hard to check in practical cases. Furthermore our result is more general in the following two senses: It applies to infinite dimensional functional differential equations, such as functional partial differential equations. Also, if the operator B generates a generalized contraction, it is always possible to change the inner product on the state space \mathcal{E}^2 such that the abstract delay semigroup is a generalized contraction semigroup.

In Section 1 we briefly describe the construction of the abstract delay semigroup, mainly to introduce notation. Then our main results are stated in Section 2. We sketch some possible applications, mainly in the field of stochastic evolution equations, in Section 3.

1 Abstract delay differential equations

We present here the abstract framework for the study of deterministic delay differential equations, or *delay equation*, of [1]. Deterministic delay equations may also be studied in spaces of continuous functions, see [7] and [8], Section VI.6, but this setting is not discussed here.

Let X, Z be Banach spaces, and let $W^{1,p}([-1, 0]; Z)$ denote the *Sobolev space* consisting of equivalence classes of functions in $L^p([-1, 0]; Z)$ which have a weak derivative in $L^p([-1, 0]; V)$ (see [10], Chapter 4).

Consider the abstract differential equation with delay

$$\begin{cases} \frac{d}{dt}u(t) = Bu(t) + \Phi u_t, & t > 0, \\ u(0) = x, \\ u_0 = f, \end{cases} \quad (3)$$

under the following assumptions:

Hypothesis 1.1. (i) $x \in X$;

(ii) B is the generator of a strongly continuous semigroup $(S(t))_{t \geq 0}$ in X ;

(iii) $\mathfrak{D}(B) \hookrightarrow Z \hookrightarrow X$ with the injections being dense;

(iv) $f \in L^p([-1, 0]; Z)$, $1 \leq p < \infty$;

(v) $\Phi : W^{1,p}([-1, 0]; Z) \rightarrow X$ is a bounded linear operator.

(vi) $u : [-1, \infty) \rightarrow X$ and for $t \geq 0$, $u_t : [-1, 0] \rightarrow X$ is defined by $u_t(\sigma) = u(t + \sigma)$, $\sigma \in [-1, 0]$.

In general, if $(\xi(t))_{t \in [-1, \infty)}$ is a process, then the process $(\xi_t)_{t \geq 0}$ with values in a function space, defined by $\xi_t(\sigma) := \xi(t + \sigma)$, $t \geq 0$, $\sigma \in [-1, 0]$, is called the *segment process* of ξ . So here $(u_t)_{t \geq 0}$ is the segment process of $(u(t))_{t \in [-1, \infty)}$. It keeps track of the history of $(u(t))_{t \in [-1, \infty)}$.

Definition 1.2. A classical solution of (3) is a function $u : [-1, \infty) \rightarrow X$ that satisfies

(i) $u \in C([-1, \infty); X) \cap C^1([0, \infty); X)$;

(ii) $u(t) \in \mathfrak{D}(B)$ and $u_t \in W^{1,p}([-1, 0]; Z)$ for all $t \geq 0$;

(iii) u satisfies (3).

To employ a semigroup approach we introduce the Banach space

$$\mathcal{E}^p := X \times L^p([-1, 0]; Z),$$

and the closed, densely defined operator in \mathcal{E}^p ,

$$A := \begin{bmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{bmatrix}, \quad \mathfrak{D}(A) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in \mathfrak{D}(B) \times W^{1,p}([-1, 0]; Z) : f(0) = x \right\}. \quad (4)$$

The equation (3) is called *well-posed* if for all $(x, f) \in \mathfrak{D}(A)$, there exists a unique classical solution of (3) that depends continuously on the initial data (in the sense of uniform convergence on compact intervals).

It is shown in [1], Corollary 3.7, that A generates a strongly continuous semigroup in \mathcal{E}^p if and only if (3) is well-posed. In this case the semigroup is called an (*abstract*) *delay semigroup*.

Furthermore, sufficient conditions on Φ are given for this to be the case:

Hypothesis 1.3. Let $S_t : X \rightarrow L^p([-1, 0]; Z)$ be defined by

$$(S_t x)(\tau) := \begin{cases} S(t + \tau)x & \text{if } -t < \tau \leq 0, \\ 0 & \text{if } -1 \leq \tau \leq -t, \end{cases} \quad t \geq 0.$$

Let $(T_0(t))_{t \geq 0}$ be the nilpotent left shift semigroup on $L^p([-1, 0]; Z)$. Assume that there exists a function $q : [0, \infty) \rightarrow [0, \infty)$ with $\lim_{t \downarrow 0} q(t) = 0$, such that

$$\int_0^t \|\Phi(S_r x + T_0(r)f)\| \, dr \leq q(t) \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\| \quad (5)$$

for all $t > 0$ and $\begin{pmatrix} x \\ f \end{pmatrix} \in \mathfrak{D}(A)$. Furthermore suppose that either

(A) $Z = X$ or

(B) (i) $(B, \mathfrak{D}(B))$ generates an analytic semigroup $(S(t))_{t \geq 0}$ on X , and

(ii) for some $\delta > \omega_0(B)$ there exists $\vartheta < \frac{1}{p}$ such that

$$\mathfrak{D}((-B + \delta I)^\vartheta) \hookrightarrow Z \hookrightarrow X,$$

with the injections being dense.

Theorem 1.4. Assume Hypothesis 1.3 holds. Then $(A, \mathfrak{D}(A))$ is the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on \mathcal{E}^p .

PROOF: See [1], Theorem 3.26 and Theorem 3.34. \square

Remark 1.5. Condition (5) is slightly stronger than needed but will also provide us with sufficient regularity of the delay semigroup later on (see [1], Section 4.1).

Example 1.6. Let $\Phi : C([-1, 0]; Z) \rightarrow X$ be given by

$$\Phi(f) := \int_{-1}^0 d\eta \, f,$$

where $\eta : [-1, 0] \rightarrow L(Z; X)$ is of bounded variation.

Suppose that either $Z = X$ or $(B, \mathfrak{D}(B))$ satisfies the assumptions (B-i) and (B-ii) of Hypothesis 1.3. Then the conditions of Hypothesis 1.3 are satisfied and hence $(A, \mathfrak{D}(A))$ generates a strongly continuous semigroup (see [1], Theorem 3.29 and Theorem 3.35). \diamond

2 Dissipativity of the delay semigroup

Recall the notions of dissipativity and contraction semigroup:

Definition 2.1. Suppose $A : \mathfrak{D}(A) \rightarrow H$ is a linear operator. Then A is said to be dissipative if $\langle Ax, x \rangle \leq 0$ for all $x \in \mathfrak{D}(A)$.

A strongly continuous semigroup $(T(t))_{t \geq 0}$ is said to be a contraction semigroup if $\|T(t)\| \leq 1$ for all $t \geq 0$.

It is well-known that a strongly continuous semigroup is a contraction semigroup if and only if the infinitesimal generator of the semigroup is dissipative.

We will continue to use the notation of the previous section. In this section we assume $X = Z$ to be a Hilbert space and $p = 2$.

Theorem 2.2. Suppose Φ has the form

$$\Phi f = \int_{-1}^0 d\zeta(s) f(s),$$

with $\zeta : [-1, 0] \rightarrow L(X)$ of bounded variation. Suppose furthermore that $B - \lambda I$ is dissipative. If there exists $\mu > \lambda$ such that

$$(\mu - \lambda)^2 > (|\zeta|(0) - |\zeta|(-1)) \int_{-1}^0 e^{2\mu r} d|\zeta|(r), \quad (6)$$

then there exists an equivalent inner product on $X \times L^2([-1, 0]; X)$ such that $A - \mu I$ is dissipative with respect to this inner product.

PROOF: Introduce a quadratic form on $X \times L^2([-1, 0]; X)$ by

$$\left(\begin{pmatrix} c \\ f \end{pmatrix}, \begin{pmatrix} d \\ g \end{pmatrix} \right) := \langle c, d \rangle + \int_{-1}^0 \tau(s) \langle f(s), g(s) \rangle ds.$$

We will make this into the new inner product satisfying the mentioned requirements. First we determine τ such that $\langle (A - \mu I)x, x \rangle \leq 0$ for all $x \in \mathfrak{D}(A)$. Then we will check that $\langle \cdot, \cdot \rangle$ is an equivalent inner product on $X \times L^2([-1, 0]; X)$.

We have for $x \in \mathfrak{D}(A)$,

$$\begin{aligned}
((A - \mu I)x, x) &= \langle Bx(0), x(0) \rangle + \left\langle \int_{-1}^0 d\zeta(s)x(s), x(0) \right\rangle \\
&\quad + \int_{-1}^0 \tau(s) \langle \dot{x}(s), x(s) \rangle ds - \mu |x(0)|^2 - \mu \int_{-1}^0 \tau(s) |x(s)|^2 ds \\
&\leq (\lambda - \mu) |x(0)|^2 + \int_{-1}^0 |x(0)| |x(s)| d|\zeta|(s) \\
&\quad + \int_{-1}^0 \tau(s) \left(\frac{1}{2} \frac{d}{ds} |x(s)|^2 - \mu |x(s)|^2 \right) ds \\
&= (\lambda - \mu) |x(0)|^2 + \int_{-1}^0 |x(0)| |x(s)| d|\zeta|(s) \\
&\quad - \int_{-1}^0 \frac{1}{2} \dot{\tau}(s) |x(s)|^2 ds + \frac{1}{2} \{ \tau(0) |x(0)|^2 - \tau(-1) |x(-1)|^2 \} \\
&\quad - \int_{-1}^0 \mu \tau(s) |x(s)|^2 ds \\
&\leq (\lambda - \mu + \frac{1}{2} \gamma) |x(0)|^2 + \int_{-1}^0 |x(0)| |x(s)| d|\zeta|(s) \\
&\quad + \int_{-1}^0 (-\mu \tau(s) - \frac{1}{2} \dot{\tau}(s)) |x(s)|^2 ds,
\end{aligned}$$

where we denote $\gamma := \tau(0)$. Since $\dot{\tau}$ may not be defined, the last steps have to be interpreted formally.

In order to be able to compare the two integrals, we demand (formally) that

$$(-\mu \tau(s) - \frac{1}{2} \dot{\tau}(s)) ds = \rho(s) d|\zeta|(s), \quad (7)$$

with $\rho : [-1, 0] \rightarrow \mathbb{R}$ some, for the moment unspecified, function.

Under this assumption, we obtain as sufficient condition for dissipativity

$$(-\mu + \lambda + \frac{1}{2} \gamma) |x(0)|^2 + \int_{-1}^0 |x(0)| |x(s)| + \rho(s) |x(s)|^2 d|\zeta|(s) \leq 0.$$

Dividing the lefthand side by $|x(0)|^2$, we obtain

$$\begin{aligned}
&(-\mu + \lambda + \frac{1}{2} \gamma) + \int_{-1}^0 \frac{|x(s)|}{|x(0)|} + \rho(s) \left(\frac{|x(s)|}{|x(0)|} \right)^2 d|\zeta(s)| \\
&= \int_{-1}^0 \frac{-\mu + \lambda + \frac{1}{2} \gamma}{|\zeta|(0) - |\zeta|(-1)} + \frac{|x(s)|}{|x(0)|} + \rho(s) \left(\frac{|x(s)|}{|x(0)|} \right)^2 d|\zeta(s)| \leq 0.
\end{aligned}$$

Note that the integrand is a polynomial in $\frac{|x(s)|}{|x(0)|}$, which is at most zero if and only if

$$\rho(s) \leq 0, \quad \text{a.a. } s \in [-1, 0],$$

and

$$1 - 4\rho(s) \frac{-\mu + \lambda + \frac{1}{2} \gamma}{|\zeta|(0) - |\zeta|(-1)} \leq 0.$$

We can obtain equality by putting $\rho(s) \equiv \rho$, with

$$\rho := \frac{|\zeta|(0) - |\zeta|(-1)}{4(-\mu + \lambda + \frac{1}{2} \gamma)}.$$

Since necessarily $\rho \leq 0$, we require

$$-\mu + \lambda + \frac{1}{2}\gamma < 0.$$

It is a straightforward exercise in partial integration to show that the expression

$$\tau(s) := e^{-2\mu s} \left[\gamma - \frac{|\zeta|(0) - |\zeta|(-1)}{-2\lambda + 2\mu - \gamma} \int_s^0 e^{2\mu r} d|\zeta|(r) \right]$$

formally solves (7), and actually satisfies the required

$$\begin{aligned} & \int_{-1}^0 \tau(s) \left(\frac{1}{2} \frac{d}{ds} |x(s)|^2 - \mu |x(s)|^2 \right) ds \\ &= \int_{-1}^0 \rho |x(s)|^2 d|\zeta|(s) + \frac{1}{2} \gamma |x(0)|^2 - \frac{1}{2} \tau(-1) |x(-1)|^2. \end{aligned}$$

For $\langle \cdot, \cdot \rangle$ to define an equivalent inner product on $X \times L^2([-1, 0]; X)$, we require that there exist constants $c_1, c_2 > 0$ such that $c_1 \leq |\tau(s)| \leq c_2$ for almost all $s \in [-1, 0]$. In particular we demand that $\tau(s) > 0$, s -a.s.

Note that $\tau(s) > 0$ if and only if

$$\gamma - \frac{|\zeta|(0) - |\zeta|(-1)}{-2\lambda + 2\mu - \gamma} \int_s^0 e^{2\mu r} d|\zeta|(r) > 0, \quad \text{a.a. } s \in [-1, 0],$$

that is

$$\gamma - \frac{|\zeta|(0) - |\zeta|(-1)}{-2\lambda + 2\mu - \gamma} \int_{-1}^0 e^{2\mu r} d|\zeta|(r) > 0$$

or equivalently

$$\gamma(-2\lambda + 2\mu - \gamma) > (|\zeta|(0) - |\zeta|(-1)) \int_{-1}^0 e^{2\mu r} d|\zeta|(r).$$

We are free to choose γ in $(0, 2(-\lambda + \mu))$, so we pick the optimal $\gamma = -\lambda + \mu$. The requirement

$$(-\lambda + \mu)^2 > (|\zeta|(0) - |\zeta|(-1)) \int_{-1}^0 e^{-2\mu r} d|\zeta|(r)$$

remains. If it is satisfied, then it is easy to check that τ is bounded from below and above and therefore does indeed define an equivalent inner product. \square

It is in general not possible to verify condition (6) explicitly. However, we may deduce the following simpler conditions.

Corollary 2.3. *Suppose Φ has the form*

$$\Phi f = \int_{-1}^0 d\zeta(s) f(s),$$

with $\zeta : [-1, 0] \rightarrow L(X)$ of bounded variation. Suppose furthermore that $B - \lambda I$ is dissipative.

(i) *Let $\mu \in \mathbb{R}$ such that*

$$\mu > \lambda + \left((|\zeta|(0) - |\zeta|(-1)) \int_{-1}^0 e^{2\lambda r} d|\zeta|(r) \right)^{1/2}. \quad (8)$$

Then the generator of the delay semigroup, A , has the property that $A - \mu I$ is dissipative.

(ii) If

$$\lambda < - \int_{-1}^0 d|\zeta|,$$

then there exists an equivalent inner product on \mathcal{E}^2 such that A is dissipative.

PROOF: The first statement follows from $e^{2\mu r} < e^{2\lambda r}$ for $\lambda < \mu$ and all $r \leq 0$, and rewriting (6). The second statement follows by noting that $\lambda^2 > (|\zeta|(0) - |\zeta|(-1))^2$ so that (6) is satisfied with $\mu = 0$. \square

In particular these conditions give us stability results for the delay semigroup. The second condition is ‘sharp’ in the sense that it reproduces the stability result [1], Corollary 5.9. For the linear delay equation consisting of two terms,

$$\dot{x}(t) = Bx(t) + Cx(t-1),$$

with $B, C \in \mathbb{R}^n$, and $B - \lambda I$ dissipative, then (8) gives the sufficient conditions $\mu > \lambda + \|C\|e^{-\lambda}$ for $A - \mu I$ to be dissipative, and $\lambda + \|C\| < 0$ in order for A to be dissipative.

To state an important consequence, we recall the notion of generalized contraction.

Definition 2.4. A strongly continuous semigroup $(T(t))_{t \geq 0}$ is called a generalized contraction if there exists a constant $\mu \in \mathbb{R}$ such that $\|T(t)\|_{t \geq 0} \leq e^{\mu t}$.

Theorem 2.5. Suppose B is the generator of a generalized contraction semigroup. Then there exists an equivalent inner product on \mathcal{E}^2 such that A is the generator of a generalized contraction semigroup.

PROOF: Denote the semigroup generated by B by $(S(t))_{t \geq 0}$ and suppose that

$$\|S(t)\| \leq e^{\lambda t}, \quad t \geq 0.$$

Let $\nu > \max\left(0, \lambda + \int_{-1}^0 d|\zeta|\right)$. Define $\tilde{B} := B - \nu I$ and $\tilde{\lambda} := \lambda - \nu$. It may be verified that the conditions of Corollary 2.3 are satisfied for \tilde{B} and $\tilde{\lambda}$, so that an equivalent inner product (\cdot, \cdot) on \mathcal{E}^2 exists such that the delay semigroup generated by

$$\tilde{A} := \begin{bmatrix} B - \nu I & \Phi \\ 0 & \frac{d}{d\sigma} \end{bmatrix}$$

is dissipative. Now for $x \in \mathfrak{D}(A)$

$$((A - \nu I)x, x) = (\tilde{A}x, x) - \nu \int_{-1}^0 \tau(\sigma)x(\sigma)^2 d\sigma \leq (\tilde{A}x, x) \leq 0.$$

\square

Note furthermore that, if A is of the form (4) with $B - \lambda I$ dissipative for some $\lambda \in \mathbb{R}$, we can always perturb A by a bounded operator of the form $\begin{pmatrix} -cI & 0 \\ 0 & 0 \end{pmatrix}$ to obtain the generator of a new delay semigroup. If we choose $c > 0$ large enough, by Corollary 2.3 we may change the inner product to obtain a dissipative generator.

In an entirely analogous way as for Theorem 2.2 we can prove the following slightly stronger result in case Φ has a density function.

Proposition 2.6. Suppose Φ is of the form

$$\Phi f = \int_{-1}^0 \zeta(\sigma)f(\sigma) d\sigma, \quad f \in L^2([-1, 0]; X),$$

with $\zeta \in L^2([-1, 0]; L(X))$. Suppose furthermore $B - \lambda I$ is dissipative. If there exists $\mu > \lambda$ such that

$$(\lambda - \mu)^2 > \int_{-1}^0 e^{2\mu\rho} \|\zeta(\rho)\|^2 d\rho.$$

then there exists an equivalent inner product on \mathcal{E}_2 such that $A - \mu I$ is dissipative with respect to this inner product.

The following is a first attempt at establishing conditions on more general generators A of strongly continuous semigroups such that there exists an equivalent inner product such that A is dissipative.

To this end we need the notion of exact observability.

Definition 2.7. Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ in a Hilbert space H . Let K be a Hilbert space and $C \in L(H; K)$. Then (A, C) is said to be exactly observable in time $\tau > 0$ if the mapping $x \mapsto C^\tau x := CT(\cdot)x : H \rightarrow L^2([0, \tau]; K)$ is injective and its inverse is bounded on the range of C^τ .

Observability is a concept dual to controllability: (A, C) is exactly observable if and only if (A, C^*) is exactly controllable (see [4], Section 4.1).

Proposition 2.8. Suppose A generates an asymptotically stable strongly continuous semigroup $(T(t))_{t \geq 0}$ in H . If there exists a Hilbert space K , a bounded linear mapping $C \in L(H; K)$ and $\tau > 0$ such that (A, C) is exactly observable in time τ , then there exists an equivalent inner product on H such that A is dissipative with respect to this inner product.

PROOF: For this C , by the characterization of exactly observable systems ([4], Corollary 4.1.14) we have

$$Q := \int_0^\infty T^*(t)C^*CT(t) dt \geq \int_0^\tau T^*(t)C^*CT(t) dt \geq \gamma I$$

for some $\gamma > 0$. Furthermore by Lyapunov theory ([4], Theorem 4.1.23) we have that $Q \in L(H)$ and

$$2\langle QAx, x \rangle = -|Cx|^2 \leq 0, \quad x \in \mathfrak{D}(A).$$

□

3 Applications

In this section we provide some examples of possible applications of the dissipativity property that we established in the previous section.

3.1 The unitary dilation theorem

An important application of dissipativity is provided by the following fundamental theorem (see [6], Theorem 7.2.1), which states that we may extend a contractive semigroup on a Hilbert space to a unitary group on a larger Hilbert space.

Theorem 3.1 (Szekőfalvi-Nagy's theorem on unitary dilations). Suppose $(T(t))_{t \geq 0}$ is a strongly continuous contraction semigroup on a Hilbert space H . Then there exists a Hilbert space K such that H is closed linear subspace of K , and a strongly continuous unitary group $(U(t))_{t \geq 0}$ such that $T(t) = PU(t)$, $t \geq 0$, where $P : K \rightarrow H$ denotes the linear projection on H .

3.2 Stochastic evolution equations

Contraction semigroups are useful when establishing existence and uniqueness of invariant measure of stochastic evolution equations. Consider the following stochastic evolution

$$\begin{cases} dX(t) = [AX(t) + F(X(t))] dt + G(X(t)) dL(t), & t \geq 0 \\ X(0) = x, \end{cases} \quad (9)$$

or in mild form

$$X(t) = T(t)x + \int_0^t T(t-s)F(X(s)) ds + \int_0^t T(t-s)G(X(s)) dW(s), \quad (10)$$

where W is a cylindrical, mean-zero Lévy process with Reproducing Kernel Hilbert Space (RKHS) \mathcal{H} , A is the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$, $F : H \rightarrow H$ and $G : H \rightarrow L_{\text{HS}}(\mathcal{H}; H)$ Lipschitz continuous (where $L_{\text{HS}}(U; H)$ denotes the Banach space of Hilbert-Schmidt operators from U into H (see [9] for definitions)).

3.2.1 Invariant measure

The following general result on the existence and uniqueness of an invariant measure relies on dissipativity of the semigroup generator.

Theorem 3.2. *Suppose there exists $\omega > 0$ such that*

$$2\langle A(x - y) + F(x) - F(y), x - y \rangle + \|G(x) - G(y)\|_{L_{\text{HS}}(\mathcal{H}; H)}^2 \leq -\omega|x - y|^2$$

for all $x, y \in H$ and $n \in \mathbb{N}$.

Then there exists exactly one invariant measure μ for (9), it is strongly mixing and there exists $C > 0$ such that for any bounded Lipschitz continuous function φ , all $t > 0$ and $x \in H$,

$$\left| P(t)\varphi(x) - \int_H \varphi d\mu \right| \leq C(1 + |x|)e^{-\omega t/2} \|\varphi\|_{\text{Lip}}.$$

PROOF: See [9], Theorem 16.5. The proof there requires the notion of Yosida approximants but it is easy to see that when $\langle Ax, x \rangle \leq \lambda|x|^2$, then for arbitrary $\varepsilon > 0$, n large enough,

$$\langle A_n x, x \rangle \leq (\lambda + \varepsilon)|x|^2,$$

where A_n denote the Yosida approximants of A . □

Combing the above theorem with the main result of this paper, we obtain the following.

Corollary 3.3. *Consider in \mathbb{R}^n the stochastic delay differential equation*

$$dx(t) = [Bx(t) + Cx(t-1) + f(x(t))] dt + g(x(t)) dW(t), \quad (11)$$

with $B, C \in \mathbb{R}^{n \times n}$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times k}$ and such that $B - \lambda$ is dissipative, f and g are globally Lipschitz (where we use the Hilbert-Schmidt norm on $\mathbb{R}^{n \times k}$), and where W is a k -dimensional Brownian motion. Suppose $\omega > 0$ is such that

$$2(\lambda + \|C\|e^{-\lambda}) + 2[f]_{\text{Lip}} + [g]_{\text{Lip}}^2 < -\omega.$$

Then there exists exactly one invariant measure on the infinite dimensional state space $\mathbb{R}^n \times L^2([-1, 0]; \mathbb{R}^n)$ for (11).

PROOF: This is an immediate result of applying Corollary 2.3, (i), with $\mu = \lambda\|C\|e^{-\lambda} + \varepsilon$ for some sufficiently small $\varepsilon > 0$ (which establishes that the corresponding delay semigroup A has the property that $A - \mu$ is dissipative), in conjunction with Theorem 3.2. □

3.2.2 Stability

It is also possible to obtain the following stability result using the dissipativity property of a delay semigroup (see [2], Section 6.6.3, and [3]).

Theorem 3.4. *Consider the one-dimensional stochastic delay differential equations*

$$dx(t) = [bx(t) + cx(t-1)] dt + \sigma x(t) dW(t),$$

with b, c and $\sigma > 0$ such that $b < \frac{1}{2}\sigma^2$ and

$$|c| < e^{-3/2\sigma^2}(\frac{1}{2}\sigma^2 - b).$$

Then the solution $(x(t))_{t \geq 0}$ is exponentially stable, almost surely.

This may be interpreted as follows: if $b < \frac{1}{2}\sigma^2$, then the equation without the delay term is almost surely exponentially stable. The second condition then gives a range for $|c|$ so that the system perturbed by an additional delay term remains stable.

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